

Particles and Antiparticles According to Covariant Schrödinger Equations

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We introduce manifestly covariant Schrödinger equations. This has the advantage that the energies of particles are always positive. Charges of particles and antiparticles have opposite signs. The physical vacuum equals the formal one. — All that occurs because manifestly covariant Schrödinger equations yield pairs of operators which differ only in some signs of eigenvalues: (i) The number of particles and the charge, (ii) the space-time wave vector and the space time momenta, (iii) the kinematical and the dynamical angular momenta.

§ 1. Manifestly Invariant Schrödinger Equations in Quantum Field Theory

Poincaré invariant lagrangeans $\mathcal{L}(x(\tau), \dot{x}(\tau))$ for a mass point at $x = (x^0, x^1, x^2, x^3) = x(\tau)$ have the homogeneity property

$$S = \int \mathcal{L}(\dot{x}, x) d\tau = \int \mathcal{L}(x, dx) . \quad (1.1)$$

In this case the parameters τ may be chosen arbitrarily, and the momenta

$$p_\mu = \partial \mathcal{L} / \partial \dot{x}^\mu \quad (1.2)$$

are not independent. There exists a relation ¹

$$K(p, x) = 0, \quad (1.3)$$

in which K is uniquely defined up to an inessential factor if the rank of the matrix $(\partial^2 \mathcal{L} / \partial \dot{x}^\mu \partial \dot{x}^\nu)$ equals 3. The above K , the so called ‘canonical function’, replaces the hamiltonian in the canonical equations:

$$\dot{p}_\mu = -\partial K / \partial x^\mu, \quad \dot{x}^\mu = +\partial K / \partial p_\mu .$$

In the free particle case,

$$\mathcal{L} = -m c^2 \sqrt{-\dot{x}^2},$$

we obtain, for instance, the well-known equations ²

$$p_\mu = \frac{m \dot{x}_\mu}{\sqrt{-\dot{x}^2}}, \quad K = p^2 + m^2 c^2 = 0 .$$

Canonical quantization in the Schrödinger picture yields immediately the operators

$$p_\mu = \frac{\hbar}{i} \frac{\partial}{\partial x^\mu} \quad (1.4)$$

and the corresponding wave equation

$$K\left(\frac{\hbar}{i} \frac{\partial}{\partial x}, x\right) \psi(x, \tau) = i \hbar \frac{\partial \psi(x, \tau)}{\partial \tau} .$$

Since K is independent of τ we may consider the τ -independent equation

$$K\left(\frac{\hbar}{i} \frac{\partial}{\partial x}, x\right) \psi(x) = \kappa \psi(x) .$$

According to (1.3) only $\kappa = 0$ should be possible. Hence, manifestly relativistic invariant Schrödinger equations may be written as

$$K\left(\frac{\hbar}{i} \frac{\partial}{\partial x}, x\right) \psi(x) = 0 . \quad (1.5)$$

Returning to the above example we obtain the usual Klein-Gordon equation

$$(-\hbar^2 \partial^\mu \partial_\mu + m^2 c^2) \psi(x) = 0 .$$

The question arises now, how manifestly relativistic invariant Schrödinger equations may be written in field theories. Up to now we only know such equations in the Heisenberg picture. If we start with classical canonical conjugate fields, say $i\bar{\psi}(x)$, $\psi(x)$, corresponding p_μ and x^μ , we miss a parameter τ . For that reason no immediate analogy is possible.

Since, as could be seen up to (1.5), τ drops out in the quantization procedure, we may extend classical theories by starting with τ -dependent canonical conjugate field functions

$$i\bar{\psi}_a(x, \tau), \quad \psi_a(x, \tau) .$$

Without any further assumption, we obtain as classical Poisson brackets:

$$\begin{aligned} [\psi_a(x, \tau), \bar{\psi}_\beta(x', \tau)] &= i \delta_{a\beta} \delta(x - x'), \\ [\psi_a(x, \tau), \psi_\beta(x', \tau)] &= [\bar{\psi}_a(x, \tau), \bar{\psi}_\beta(x', \tau)] = 0 \end{aligned} \quad (1.6)$$

with the 4-dimensional δ -function. Denoting the canonical function

$$iK(\bar{\psi}, \psi) ,$$



where K is a given functional of the fields, we obtain

$$\dot{\psi}(x, \tau) = +\delta K / \delta \bar{\psi}(x, \tau), \dot{\bar{\psi}}(x, \tau) = -\delta K / \delta \psi(x, \tau)$$

as canonical equations. The usual wave equations occur again if we look for τ -independent solutions only. So we have a legitimate extension of the classical theory.

Canonical quantization in the Schrödinger picture requires τ -independent operators $\bar{\psi}(x)$, $\psi(x)$ satisfying, in the fermion case, commutation relations with plus-brackets corresponding to (1.6). They read, using $\hbar = 1$:

$$\begin{aligned} \{\psi_\alpha(x), \bar{\psi}_\beta(x')\} &= \delta_{\alpha\beta} \delta(x-x'), \\ \{\psi_\alpha(x), \psi_\beta(x')\} &= \{\bar{\psi}_\alpha(x), \bar{\psi}_\beta(x')\} = 0. \end{aligned} \quad (1.7)$$

With the arguments leading to (1.5) the Schrödinger operators $K(\bar{\psi}, \psi)$ yield the covariant and adjoint Schrödinger equations

$$K(\bar{\psi}, \psi) |\Phi\rangle = 0, \langle\Phi| K(\bar{\psi}, \psi) = 0, \quad (1.8)$$

where $|\Phi\rangle$ and $\langle\Phi|$ are KET- and BRA-vectors still to be defined. In particular we consider the Schrödinger operator for Dirac fermions,

$$K = \int d^4x \bar{\psi}(x) (\gamma^\mu \partial_\mu + m) \psi(x), \quad (1.9)$$

because it leads to some unexpected results. We use the notation

$$\begin{aligned} K &= \int d^4x_1 d^4x_2 \bar{\psi}(x_1) D(x_1 - x_2) \psi(x_2), \\ D(x) &= \gamma^\mu \partial_\mu \delta(x) + m \delta(x), \end{aligned} \quad (1.10)$$

in which no derivatives of operators occur. The integral is understood only as a linear combination of the operator products $\bar{\psi}_\alpha(x') \psi_\beta(x'')$. In particular, no recipe is given to compute operator integrals and no one is necessary.

The solutions of the adjoint Schrödinger equations in (1.7) are well-defined if we start with the adjoint representations of the vacuum, namely the KET vacuum $|0\rangle$, and the BRA vacuum $\langle 0|$, which by definition satisfy the conditions

$$\psi_\alpha(x) |0\rangle = 0, \langle 0| \bar{\psi}_\alpha(x) = 0, \langle 0| 0\rangle = 1. \quad (1.11)$$

Obviously, both vacua are solutions of (1.7):

$$K |0\rangle = 0, \langle 0| K = 0. \quad (1.12)$$

The 1-particle-vectors

$$\begin{aligned} |\Phi\rangle &= \int d^4x \bar{\psi}(x) \varphi(x) |0\rangle, \\ \langle\Phi| &= \langle 0| \int d^4x \bar{\varphi}(x) \psi(x) \end{aligned} \quad (1.13)$$

are solutions if

$$\begin{aligned} K |\Phi\rangle &= \int d^4x_1 d^4x_2 d^4x \bar{\psi}(x_1) D(x_1 - x_2) \\ &\quad \cdot \psi(x_2) \bar{\psi}(x) \varphi(x) |0\rangle \\ &= \int d^4x_1 d^4x_2 \bar{\psi}(x_1) D(x_1 - x_2) \varphi(x_2) |0\rangle = 0 \end{aligned}$$

and

$$\begin{aligned} \langle\Phi| K &= \langle 0| \int d^4x d^4x_1 d^4x_2 \\ &\quad \cdot \bar{\varphi}(x) \psi(x) \bar{\psi}(x_1) D(x_1 - x_2) \psi(x_2) \\ &= \langle 0| \int d^4x_1 d^4x_2 \bar{\varphi}(x_1) D(x_1 - x_2) \psi(x_2) = 0. \end{aligned}$$

Since the KET-vectors $\bar{\psi}_\alpha(x) |0\rangle$ are linearly independent and also the BRA-vectors $\langle 0| \psi_\alpha(x)$, we obtain the adjoint wave equations of Dirac:

$$\begin{aligned} \int d^4x' D(x - x') \varphi(x') d^4x' &\equiv (\gamma^\mu \partial_\mu + m) \varphi(x) = 0, \\ \int d^4x' \bar{\varphi}(x') D(x' - x) &\equiv \bar{\varphi}(x) (-\gamma^\mu \bar{\partial}_\mu + m) = 0, \end{aligned} \quad (1.14)$$

in which ∂_μ and $\bar{\partial}_\mu$ denote the right-hand- and the left-hand-side derivatives $\partial/\partial x^\mu$.

The adjoint vectors span two different vector spaces even if we consider one-particle states only, namely the KET-, and the BRA-space. The vector in both spaces are contragredient as in affine geometry. The scalar product,

$$\begin{aligned} \langle\Phi| \Phi'\rangle &= \int d^4x d^4x' \langle 0| \bar{\varphi}(x) \psi(x) \bar{\psi}(x') \varphi'(x') |0\rangle \\ &= \int d^4x \bar{\varphi}(x) \varphi'(x) \end{aligned}$$

namely, is well defined by the commutation relations and the vacuum conditions, including $\langle 0| 0\rangle = 1$, and invariant under contragredient linear transformations. This refers here of course only to the transformations of the one-particle states.

The wave functions $\bar{\varphi}(x)$ and $\varphi'(x)$ are independent solutions of the adjoint wave equations. Within the framework of affine geometry there is no connection between them. However, such a relation must be found in physics. Since the vectors in each space are related to the same set of one-particle states, there must exist adjoint pairs of wave functions $\bar{\varphi}(x)$ and $\varphi(x)$ belonging to the same state.

These pairs are well-known and uniquely determined up to a c -number factor. Starting with the first equation in (1.14) the hermitean conjugate equation reads

$$\varphi^+(x) (\gamma^{\mu*} \bar{\partial}_\mu + m) = 0.$$

If the signature of the metric equals $-+++$, the following representation of the γ -matrices is possible:

$$\begin{aligned} \gamma^0 &= -i \varrho_3, \quad (\gamma^1, \gamma^2, \gamma^3) = \varrho_2 \sigma, \quad \gamma^5 = \varrho_1, \\ \gamma^{\alpha\beta} &= \frac{1}{2i} [\gamma^\alpha, \gamma^\beta], \quad \alpha, \beta \in (0, 1, 2, 3, 5), \end{aligned} \quad (1.15)$$

with hermitean, and antisymmetric 4×4 -matrices ϱ and σ which satisfy the commutation relations:

$$\varrho \times \varrho = 2i\varrho, \quad \sigma \times \sigma = 2i\sigma, \quad [\varrho_i, \sigma_k] = 0. \quad (1.16)$$

According to these relations we obtain

$$\gamma^{\mu+} = -\varrho_3 \gamma^{\mu} \varrho_3 \quad (1.17)$$

and

$$\varphi^+(x) \varrho_3 (-\gamma^{\mu} \partial_{\mu} + m) = 0.$$

Hence the adjoint solution equals

$$\bar{\varphi}(x) = \varphi^+(x) \varrho_3 \varepsilon, \quad \varepsilon = \pm 1, \quad (1.18)$$

where ε may be an arbitrary c -number constant. Since $\varphi(x)$ contains again such a constant, ε may be restricted to ± 1 without loss of generality.

Till now there has been no doubt that ε must equal ± 1 . That is a consequence of Schrödinger equations like

$$H|\Phi\rangle = i|\dot{\Phi}\rangle,$$

which are not yet written manifestly covariant. Since solutions with arbitrary initial conditions are possible, the norm $\langle \Phi | \Phi \rangle$ must be positive for all vectors. Hence, the metric form must be positive definite.

According to (1.8) manifest covariant Schrödinger equations yield vectors of a certain subspace. Only the norms of subspace vectors must be positive for physical reasons. Hence, the metric form may be indefinite. If no vector $|\Phi\rangle \neq 0$ in the subspace has zero norm, a sign factor is sufficient to make all occurring norms positive. That may be the factor ε introduced above.

According to (1.13, 18) we obtain for any pair of adjoint solutions of the Dirac equations the expression

$$\langle \Phi | \Phi \rangle = \varepsilon \int d^4x \varphi^+(x) \varrho_3 \varphi(x) \quad (1.19)$$

as the square of the norm. The integral is obviously indefinite. The factor ε may be chosen so that (1.19) is at least semidefinite. We shall prove now that this metric is actually positive definite.

§ 2. Solutions of the Dirac Equations

To compute the norm we consider the plane wave solutions⁴ of (1.14):

$$\varphi(x) = u e^{ik \cdot x}. \quad (2.1)$$

The amplitudes u must satisfy the equation

$$(i \gamma \cdot k + m)u = 0. \quad (2.2)$$

Hence

$$u = (m - i \gamma \cdot k)v, \quad k^2 = -m^2,$$

with arbitrary spinors v . With

$$k \equiv (k^0, \mathbf{k}) = (\varepsilon \omega, \mathbf{k}), \quad \omega = \sqrt{\mathbf{k}^2 + m^2}, \quad \varepsilon = \pm 1, \quad (2.3)$$

the norm will become positive if the sign of the frequency is given by the ε in (1.18). Separating space and time components we may rewrite u as

$$u = (m + \varepsilon \omega \varrho_3 - i \varrho_2 \sigma \cdot \mathbf{k})v. \quad (2.4)$$

Since $\sigma \cdot \mathbf{k}$ commutes with the matrix in (2.2), we may introduce the eigenvalues

$$\sigma \cdot \mathbf{k} v = \varepsilon' k v, \quad k = |\mathbf{k}|, \quad \varepsilon' = \pm 1. \quad (2.5)$$

Hence

$$v = (1 + \varepsilon' \mathbf{e} \cdot \sigma)w, \quad \mathbf{e} = \mathbf{k}/k. \quad (2.6)$$

Inserting that into (2.4) we obtain

$$u = (m + \varepsilon \omega \varrho_3 - i \varepsilon' k \varrho_2) (1 + \varepsilon' \mathbf{e} \cdot \sigma)w. \quad (2.7)$$

Since the spinor w is again arbitrary, we may replace w by

$$(1 + \varepsilon'' \varrho_3)w, \quad \varepsilon'' = \pm 1.$$

Hence, we obtain finally⁵

$$u = (m + \varepsilon \omega \varrho_3 - i \varepsilon' k \varrho_2) (1 + \varepsilon' \sigma \cdot \mathbf{e}) (1 + \varepsilon'' \varrho_3)w. \quad (2.8)$$

The signs $\varepsilon, \varepsilon', \varepsilon''$ are certain quantum numbers. As we shall see later on, ε has something to do with the charge, and ε' with the helicity. The sign ε'' is not the usual, but a kind of parity. Inversion I ,

$$u(\omega, \mathbf{k}) \rightarrow \varrho_3 u(\omega, -\mathbf{k}), \quad (2.9)$$

yields

$$u' = \varrho_3 (m + \varepsilon \omega \varrho_3 - i \varepsilon' k \varrho_2) (1 - \varepsilon' \sigma \cdot \mathbf{e}) (1 + \varepsilon'' \varrho_3)w \\ = \varepsilon'' (m + \varepsilon \omega \varrho_3 + i \varepsilon' k \varrho_2) (1 - \varepsilon' \sigma \cdot \mathbf{e}) (1 + \varepsilon'' \varrho_3)w.$$

Adding a spin flip, $F, \varepsilon' \rightarrow -\varepsilon'$, we obtain

$$u'' = \varepsilon'' (m + \varepsilon \omega \varrho_3 - i \varepsilon' k \varrho_2) \\ \cdot (1 + \varepsilon' \sigma \cdot \mathbf{e}) (1 + \varepsilon'' \varrho_3)w = \varepsilon'' u.$$

Hence

$$F I u = \varepsilon'' u. \quad (2.10)$$

Certainly, the parity itself is a constant of motion. However, I does not commute with $\sigma \cdot \mathbf{k}$. Therefore, I and $\sigma \cdot \mathbf{e}$ are not good quantum numbers at the same time. According to (2.10) I is replaced by the quantum number $F I = \varepsilon''$. We give still another,

more explicite proof for that. The flip of $\mathbf{e} \cdot \boldsymbol{\sigma}$ is produced by $F = \mathbf{a} \cdot (\mathbf{e} \times \boldsymbol{\sigma})$, for

$$F \mathbf{e} \cdot \boldsymbol{\sigma} = -\mathbf{e} \cdot \boldsymbol{\sigma} F.$$

Hence, as in (2.10) we obtain

$$F I \mathbf{e} \cdot \boldsymbol{\sigma} = -F \mathbf{e} \cdot \boldsymbol{\sigma} I = \mathbf{e} \cdot \boldsymbol{\sigma} F I.$$

Using (2.8) we obtain the expression

$$\langle \Phi | \Phi \rangle = +\varepsilon \Omega w^+ (1 + \varepsilon'' \varrho_3) (1 + \varepsilon' \mathbf{e} \cdot \boldsymbol{\sigma}) (m + \varepsilon \omega \varrho_3 + i \varepsilon' k \varrho_2) \cdot \varrho_3 (m + \varepsilon \omega \varrho_3 - i \varepsilon' k \varrho_2) (1 + \varepsilon' \mathbf{e} \cdot \boldsymbol{\sigma}) (1 + \varepsilon'' \varrho_3) w.$$

Because of

$$(1 + \varepsilon' \mathbf{e} \cdot \boldsymbol{\sigma})^2 = 2(1 + \varepsilon' \mathbf{e} \cdot \boldsymbol{\sigma}),$$

$$\begin{aligned} \varrho_3 (m + \varepsilon \omega \varrho_3 - i \varepsilon' k \varrho_2)^2 &= 2 \varepsilon m \omega \left(1 + \varepsilon \frac{m}{\omega} \varrho_3 - \varepsilon \varepsilon' \frac{k}{\omega} \varrho_1 \right), \\ (1 + \varepsilon'' \varrho_3)^2 &= 2(1 + \varepsilon'' \varrho_3), \quad (1 + \varepsilon'' \varrho_3) \varrho_1 (1 + \varepsilon'' \varrho_3) = 0, \end{aligned}$$

we get norms

$$\langle \Phi | \Phi \rangle = 8 m \omega \Omega w^+ (1 + \varepsilon'' \varrho_3) (1 + \varepsilon \boldsymbol{\sigma} \cdot \mathbf{e}) \left[1 + \varepsilon \left(\frac{m}{\omega} \right) \varrho_3 \right] w > 0. \quad (2.12)$$

which are positive definite: At least two of the matrix factors are semidefinite. So zero norms seem to be possible. However, if the norm is zero, one of the following relations must occur:

$$(1 + \varepsilon'' \varrho_3) w = 0, \quad (1 + \varepsilon' \mathbf{e} \cdot \boldsymbol{\sigma}) w = 0, \quad \left[1 + \varepsilon \frac{m}{\omega} \varrho_3 \right] w = 0.$$

In all these cases the amplitude u will also vanish. Hence, solutions of that kind do not exist. The norm is always positive due to the fact that the sign factor ε appears twice, once from (1.18), once as frequency sign.

§ 3. Integrals of Motion

According to symmetry properties the following integrals commute with the Schrödinger-Operator K :

$$\begin{aligned} N &= \int d^4x \bar{\psi}(x) \psi(x), \\ K_\mu &= \int d^4x_1 d^4x_2 \bar{\psi}(x_1) \nabla_\mu(x_1 - x_2) \psi(x_2), \\ L^{\mu\nu} &= \int d^4x_1 d^4x_2 \bar{\psi}(x_1) A^{\mu\nu}(x_1 - x_2) \psi(x_2). \end{aligned} \quad (3.1)$$

Again we consider the operator integrals only as linear combinations of operator products. The kernels read:

$$\begin{aligned} \nabla_\mu(x) &= i \partial_\mu \delta(x), \\ A^{\mu\nu}(x) &= (-i(x^\mu \partial^\nu - x^\nu \partial^\mu) + \frac{1}{2} \gamma^{\mu\nu}) \delta(x). \end{aligned} \quad (3.2)$$

Proof: Operators of the above kind, say

$$\begin{aligned} A &= \int d^4x_1 d^4x_2 \bar{\psi}(x_1) \hat{A}(x_1 - x_2) \psi(x_2), \\ B &= \int d^4x_1 d^4x_2 \bar{\psi}(x_1) \hat{B}(x_1 - x_2) \psi(x_2) \end{aligned}$$

commute, if

$$[A, B] = \int d^4x_1 d^4x_2 \bar{\psi}(x_1) [\hat{A}, \hat{B}]_{x_1 x_2} \psi(x_2) = 0.$$

Hence

$$[\hat{A}, \hat{B}]_{x_1 x_2} = \int d^4x (\hat{A}(x_1 - x) \hat{B}(x - x_2) - \hat{B}(x_1 - x) \hat{A}(x - x_2)) = 0.$$

These equations yields

$$[K, N] = 0, \quad [K, K_\mu] = 0, \quad [K, L^{\mu\nu}] = 0 \quad (3.3)$$

for

$$[\gamma^e \partial_e + m, 1] = 0, \quad [\gamma^e \partial_e + m, -i \partial_\mu] = 0, \\ [\gamma^e \partial_e + m, -i(x^\mu \partial^\nu - x^\nu \partial^\mu) + \frac{1}{2} \gamma^{\mu\nu}] = -i(\gamma^\mu \partial^\nu - \gamma^\nu \partial^\mu) + i(\gamma^\mu g^{\nu e} - \gamma^\nu g^{\mu e}) \partial_e = 0.$$

The last equations are derived by using the second of the commutators for γ -matrices:

$$[\gamma^\mu, \gamma^\nu] = 2i \gamma^{\mu\nu}, \quad [\gamma^e, \gamma^{\mu\nu}] = 2i(\gamma^\mu g^{\nu e} - \gamma^\nu g^{\mu e}), \\ [\gamma^{\mu\nu}, \gamma^{\rho\sigma}] = 2i(\gamma^{\rho\nu} g^{\sigma\mu} - \gamma^{\sigma\nu} g^{\rho\mu} + \gamma^{\mu\rho} g^{\sigma\nu} - \gamma^{\mu\sigma} g^{\rho\nu}). \quad (3.4)$$

They are valid even for the index 5, cf. (1.15).

The operators (3.1) define, as usual, infinitesimal transformations, namely the phase transformation

$$\delta\psi = +i\psi, \quad \delta\bar{\psi} = -i\bar{\psi}, \quad (3.7)$$

the translations

$$\delta\psi = +iK_\mu \psi, \quad \delta\bar{\psi} = -\bar{\psi} K_\mu, \quad (3.8)$$

and the Lorentz transformations

$$\delta\psi = +iL^{\mu\nu} \psi, \quad \delta\bar{\psi} = -i\bar{\psi} L^{\mu\nu}. \quad (3.9)$$

They are in any case contragredient, and unitary as far as the kernels are hermitean. Hermiticity is satisfied in (3.7/8) and for

$$\mathbf{L} = (L^{23}, L^{31}, L^{12}), \quad (3.10)$$

but not for

$$\mathbf{L}^0 = (L^{01}, L^{02}, L^{03}) \quad (3.11)$$

because

$$\gamma^0 \equiv (\gamma^{01}, \gamma^{02}, \gamma^{03}) = -i\varrho_1 \sigma \quad (3.12)$$

is skew hermitean. However, Lorentz invariance is not violated by that, and the problem of indefinite metric is overcome, as we have seen.

Since all components of K_μ commute with each other and with N , simultaneous eigensolutions exist. They are represented by (2.1). We obtain immediately

$$N|\Phi\rangle = |\Phi\rangle, \quad K_\mu|\Phi\rangle = k_\mu|\Phi\rangle. \quad (3.13)$$

The eigenvalue of N equals the particle number, and those of K_μ define the wave vector. They are different from charge and space-time momenta.

Since the operators $\Lambda^{\mu\nu}$ do not commute with all $\Lambda^{e\sigma}$ and K_e :

$$[\nabla^e, \Lambda^{\mu\nu}] = i(\nabla^\mu g^{\nu e} - \nabla^\nu g^{\mu e}) \neq 0 \quad (3.14)$$

$$[\Lambda^{\mu\nu}, \Lambda^{e\sigma}] = i(\Lambda^{e\nu} g^{\sigma\mu} - \Lambda^{e\sigma} g^{\mu e} + \Lambda^{\mu e} g^{\sigma\nu} - \Lambda^{\mu\sigma} g^{\nu e}) \neq 0 \quad (3.15)$$

simultaneous eigensolutions exist only for the spin operator

$$\Sigma_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \nabla^\nu \Lambda^{e\sigma} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \nabla^\nu \gamma^{e\sigma}, \quad (3.16)$$

where $\varepsilon_{\mu\nu\rho\sigma}$ equals +1 if the indices are even permutations of 0123, equals -1 if they are odd permutations, and equals 0 in all other cases. Space-time separation yields

$$\Sigma_0 = -\frac{i}{2} \sigma \cdot \nabla \delta(x), \quad \Sigma = -\frac{i}{2} (\sigma \partial_0 - \varrho_1 \sigma \times \nabla).$$

All components of (Σ_μ) commute with all of (∇_μ) . Only certain pairs of $\Lambda^{\mu\nu}$ commute, for instance Λ^{01} with Λ^{23} . Hence we expect two linear combinations of Σ_μ , which commute. However one of them vanishes, namely

$$\nabla^\mu \Sigma_\mu = 0.$$

Therefore only one component remains, anyone. Usually⁶ the helicity is considered, which is connected with

$$\Lambda_0(x) = -\frac{i}{2} \sigma \cdot \nabla \delta(x)$$

and

$$L_0 = \int d^4x_1 d^4x_2 \bar{\psi}(x_1) \Lambda_0(x_1 - x_2) \psi(x_2). \quad (3.17)$$

Applying this operator on the 1-particle state vector, defined by (2.8), we obtain

$$\Lambda_0|\Phi\rangle = \frac{1}{2} \varepsilon' k |\Phi\rangle. \quad (3.18)$$

The constant

$$\lambda_h = \frac{1}{2} \varepsilon' = \pm \frac{1}{2} \quad (3.19)$$

is the helicity.

So far all results are more or less well known. Completely new results appear, if we start with the commutators

$$[K, \psi(x)] = -(\gamma^\mu \partial_\mu + m) \psi(x), \\ [K, \bar{\psi}(x)] = -\bar{\psi}(x) (\gamma^\mu \bar{\partial}_\mu - m).$$

The derivatives are defined again by

$$\begin{aligned}\partial_\mu \psi(x) &= \int d^4x' \partial_\mu \delta(x-x') \psi(x'), \\ \partial_\mu \bar{\psi}(x) &= \int d^4x' \partial_\mu \delta(x'-x) \bar{\psi}(x').\end{aligned}$$

Left-hand-side multiplication of the first equation with $\bar{\psi}(x)$, right-hand-side multiplication of the second one with $\psi(x)$ and addition of both yield

$$[K, \bar{\psi}(x) \psi(x)] = \partial_\mu (\bar{\psi}(x) \gamma^\mu \psi(x)).$$

Introducing two arbitrary solutions $|\Phi\rangle$ and $|\Phi'\rangle$ of the Schrödinger equation, we obtain the continuity equation

$$\partial_\mu \langle \Phi | \bar{\psi}(x) \gamma^\mu \psi(x) | \Phi' \rangle = 0.$$

Hence, the space integrals of the time component are constant,

$$\int \langle \Phi | \bar{\psi}(x) \varrho_3 \psi(x) | \Phi' \rangle d^3\mathbf{r} = \text{const}, \quad (3.20)$$

if the flux through the space surface vanishes.

Since creations and annihilations take place in space-time points, probabilities are defined only in space-time volumes. Space volumes have the measure zero. Therefore a further integration is necessary to obtain relevant results from (3.20).

If we return for a while to the τ -dependent Schrödinger equations we can see easily what we must do: We have

$$i |\dot{\Phi}'\rangle = K |\Phi'\rangle, \quad -i \langle \Phi | = \langle \Phi | K.$$

and in this case we obtain from

$$\langle \Phi | [K, \bar{\psi} \psi] | \Phi' \rangle = \partial_\mu \langle \Phi | \bar{\psi} \gamma^\mu \psi | \Phi' \rangle$$

the five-dimensional continuity equation

$$-i \partial_\tau \langle \Phi | \bar{\psi} \psi | \Phi' \rangle = \partial_\mu \langle \Phi | \bar{\psi} \gamma^\mu \psi | \Phi' \rangle.$$

Hence, the five-dimensional volume integration yields

$$\int \langle \Phi | \bar{\psi}(x) \varrho_3 \psi(x) | \Phi' \rangle d\tau d^3\mathbf{r} = \text{const} \quad (3.21)$$

instead of (3.20), if the flux through the space surfaces vanishes as above, and if we return to τ -independent solutions.

In space-time symmetrical canonical mechanics the parameter τ is defined by K . Here, that should be true too. To find τ we start with the Dirac equation

$$(\gamma^\mu \partial_\mu + m) \varphi(x) = 0$$

and consider two solutions $\varphi(x)$ and $\varphi'(x)$, one valid in a certain space-time domain A, the other in B. Both domains may have a common boundary

$$S'(x) = 0.$$

We assume⁷ that both solutions are continuously connected at the boundary, and ask for possible

jumps of the derivatives. Since tangential derivatives must be continuous, the jumps may be written as

$$[\varphi(x)] = 0, \quad [\partial_\mu \varphi(x)] = \eta(x) \partial_\mu S(x).$$

Hence, we obtain the rigorously valid condition

$$\gamma^\mu \partial_\mu S(x) \eta(x) = 0$$

from the Dirac equation. According to that, real jumps are only possible, if

$$\partial^\mu S(x) \partial_\mu S(x) = 0.$$

That is a Hamilton-Jacobian differential equation belonging to the canonical function

$$K = -\sqrt{-p^2} = 0,$$

where the square root corresponds in the sense of Dirac to

$$-i \gamma^\mu \partial_\mu \sim \sqrt{-p^2}.$$

The canonical equations

$$\dot{x}^\mu = p^\mu / \sqrt{-p^2}, \quad \dot{p}_\mu = 0$$

yield

$$\dot{x}^\mu x_\mu = -1, \quad d\tau = \sqrt{-dx^2} = \sqrt{1 - \beta^2} dx^0 = (m/\omega) dx^0.$$

So we obtain finally from (3.21):

$$(m/\omega) \int \langle \Phi | \bar{\psi}(x) \varrho_3 \psi(x) | \Phi' \rangle d^4x = \text{const}. \quad (3.22)$$

Since $|\Phi'\rangle$ should be a solution of the Schrödinger equation like $|\Phi\rangle$, we may introduce according to (3.1 and 1.7):

$$|\Phi'\rangle = |\Phi\rangle, \quad K_\mu |\Phi\rangle, \quad L_0 |\Phi\rangle. \quad (3.23)$$

The first solution yields the integral

$$Q = (m/\omega) \int \langle \Phi | \bar{\psi}(x) \varrho_3 \psi(x) | \Phi \rangle d^4x.$$

That has the same structures than the charge in conventional quantum field theory. The 1-particle-solution leads to the expression

$$Q = (m/\omega) \Omega \bar{u} \varrho_3 u = (\varepsilon m/\omega) \Omega u^+ u.$$

According to (2.8) we obtain

$$\begin{aligned}u^+ u &= w^+ (1 + \varepsilon'' \varrho_3) (1 + \varepsilon' \boldsymbol{\sigma} \cdot \mathbf{e}) (m + \varepsilon \omega \varrho_3 + i \varepsilon' k \varrho_2) \\ &\quad (m + \varepsilon \omega \varrho_3 - i \varepsilon' k \varrho_2) \cdot (1 + \varepsilon' \boldsymbol{\sigma} \cdot \mathbf{e}) (1 + \varepsilon'' \varrho_3) w.\end{aligned}$$

That equals, using (2.12),

$$\begin{aligned}u^+ u &= 8 \omega^2 w^+ (1 + \varepsilon'' \varrho_3) (1 + \varepsilon' \boldsymbol{\sigma} \cdot \mathbf{e}) (1 + \varepsilon(m/\omega) \varrho_3) w \\ &= (\omega/m \Omega) \langle \Phi | \Phi \rangle = \omega/m \Omega.\end{aligned}$$

Hence

$$Q = \varepsilon. \quad (3.24)$$

The charge equals ± 1 . The upper sign belongs to positive frequencies, that means to particles, the lower sign to antiparticles (or vice versa). Particles and antiparticles have opposite charges.

The second solution in (3.23) yields the integrals

$$P_\mu = (-im/\omega) \int \langle \Phi | \bar{\psi}(x) \partial_\mu \psi(x) | \Phi \rangle d^4x \quad (3.25)$$

for the space-time momenta. As above we obtain

$$P_\mu = \varepsilon k_\mu. \quad (3.26)$$

Momenta of particles are given by the wave vector, of antiparticles by the opposite one. In particular, the energy

$$W \equiv P^0 = \varepsilon k^0 = \omega = \sqrt{m^2 + \mathbf{k}^2} > 0 \quad (3.27)$$

is always positive.

The author proposed many years ago⁸ a sign factor ε because the wave vector k^μ is, according to the invariance of $\exp(i k x)$, a common space-time vector, while $p^0 > 0$ shows that p^μ must be a kind of pseudovector. Now, the result is satisfying. The sign factor appears without any hypothesis, if we consider manifest covariant Schrödinger equations, and the subspace given by $K | \Phi \rangle = 0$.

Finally we control the helicity counterpart:

$$H_0 = \frac{-im}{2\omega} \int \langle \Phi | \bar{\psi}(x) \partial_3 \psi(x) | \Phi \rangle d^4x. \quad (3.28)$$

The result is similar to that in (3.18). Only $k = |\mathbf{k}|$ is replaced by $p = |\mathbf{p}|$:

$$H_0 | \Phi \rangle = \frac{1}{2} \varepsilon' p | \Phi \rangle. \quad (3.29)$$

The quantum number (3.19) of the helicity remains unchanged, if we define

$$\lambda_h = L_0/k = H_0/p. \quad (3.30)$$

We cannot but state that the change is a very deep one if we start with the manifestly covariant Schrödinger equation. The sign problem, which Dirac has solved so ingeniously, has vanished. Furtheron we need neither holes, nor vacua depending on interactions. The so-called 'formal' vacuum may be the 'physical' one. At least, there are no longer traditional reasons to introduce other vacua.

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¹ F. Bopp, Eine Spinorfeldtheorie im explizite relativistisch invarianten Schrödinger-Bild, Sitz.-Ber. Bay. Akad. Wiss. Math.-Naturw. Kl. (1975), im Druck.

² D. Ter Haar, Elements of Hamiltonian Mechanics, North-Holland Publ. Cy., Amsterdam 1961, Ch. 5, § 4, Eq's. (5.429, 436). — J. L. Synge, Classical Dynamics, Sect's. 64, 86, in Encyclopedia of Physics, edited by S. Flügge, Vol. III/1, Springer-Verlag, Berlin 1960.

³ The right-hand-side of the commutation relations is, of course, not a solution of any wave equation in the Schrödinger picture.

⁴ Any text book containing relativistic quantum mechanics, e.g. P. Roman, Theory of Elementary Particles, North-Holland Publ. Cy., Amsterdam 1961, hier Chapter II.

⁵ The last matrix factor does not occur in most textbooks.

⁶ S. S. Schweber, An Introduction to Relativistic Quantum Field Theory, Harper & Row Publishers, New York 1962, p. 83.

⁷ G. Herglotz, Mechanik der Kontinua, Göttingen lecture 1931/32, Library of the Mathematical Institute, University of Göttingen. — C. Truesdell and R. Toupin, The Classical Field Theories, Sect. 175, pp. 493–496, in Encyclopedia of Physics, edited by S. Flügge, Vol. III/1, Springer-Verlag, Berlin 1960.

⁸ F. Bopp, Sitz.-Ber. Bay. Akad. Wiss., Math.-Naturw. Kl. (1964), S. 71 ff.